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Technical Notes

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Tension Buckling in Shear-Flexible Composite Beams

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Introduction

It was recently reported [1] that short, shear-flexible beams made of rubberlike materials can buckle in tension. It is the purpose of this Note to show that a similar phenomenon can occur in certain composite beams. In order for a composite beam to exhibit the buckling phenomenon described herein, it must have a shearing stiffness that is much less than the extensional stiffness. This can happen, for example, in so-called circumferentially uniform stiffness composite beams, which can be manufactured by filament winding, as discussed by Rehfield et al. [2]. In such beams the shearing stiffness can be as much as 2 orders of magnitude smaller than the extensional stiffness [3]. It is not required that the beam be short, however.

Kinematics and Potential of Applied Load

The analysis is based on a consistent approximation of geometrically exact theory [4]. A planar version of this theory [5] is sufficient for the present discussion. The longitudinal and shear force strains are given by

$$\begin{aligned} \epsilon &= (1+u')\cos\theta + v'\sin\theta - 1 \\ \gamma &= -(1+u')\sin\theta + v'\cos\theta \end{aligned} \tag{1}$$

where we have denoted the longitudinal displacement as u, the transverse displacement as v, and the section rotation as θ. The moment strain associated with beam bending is simply θ'. The derivative with respect to x is denoted by $()'$. In this theory, as in the conventional Timoshenko beam theory, the section rotation and displacement variables are independent.

The moment strain is already a linear quantity. A linear approximation of the force strain associated with transverse shear is sufficient for the present purposes and can be written as $\gamma = v' - \theta$. However, if the beam is loaded in the axial direction by a force, the potential energy of that force must be derived from *the quadratic terms in the longitudinal strain measure*. We can approximate the first of Eqs. (1) to second order by

$$\epsilon = \bar{\epsilon} + v'\theta - \tfrac{1}{2}\theta^2 + \dots \tag{2}$$

where the dots refer to terms of third and higher degree in the unknowns. The first term represents the stretching strain associated with the prebuckling state, such that $\bar{\epsilon} = P/EA$, with positive P representing an axial tensile force and EA the axial rigidity. Taking the strain energy per unit length contributed only by deformation dependent terms from the axial force, we thus have

$$V = \frac{EA\epsilon^2}{2} = \frac{P}{2}(2v'\theta - \theta^2) \quad \text{shear flexible approximation of longitudinal strain} \tag{3}$$

This is different from the usual expression for this term, which is obtained by setting $\theta = v'$ in Eq. (3), so that

$$\bar{V} = \frac{EA\epsilon^2}{2} = \frac{Pv'^2}{2} \quad \text{shear rigid approximation of longitudinal strain} \tag{4}$$

It will turn out that the phenomenon under consideration can only be predicted if Eq. (3) is used for the potential of the applied load P.

Buckling of Composite Beams Under Tension

Consider a spanwise uniform, shear-flexible, cantilevered beam loaded by an axial force P. The total potential energy, when specialized for deformation in a plane and after all terms of third and higher degree in the unknown displacement v and section rotation θ are dropped, reduces to

$$U + V = \frac{1}{2}\int_0^\ell [\mathrm{EI}\theta'^2 + \mathrm{GK}(v'-\theta)^2 + P(2v'\theta - \theta^2)]\,\mathrm{d}x \tag{5}$$

where U is the strain energy, EI is the bending stiffness, and GK is the transverse shearing stiffness. Minimization of $U + V$ subject to the essential boundary conditions that v and θ vanish at $x = 0$ provides the governing differential equations and natural boundary conditions

$$\begin{gathered} \mathrm{EI}\,\theta'' + (\mathrm{GK} - P)(v' - \theta) = 0 \qquad \mathrm{GK}(v'-\theta)' + P\theta' = 0 \\ \mathrm{EI}\theta'(\ell) = \mathrm{GK}[v'(\ell) - \theta(\ell) + P\theta(\ell) = 0 \end{gathered} \tag{6}$$

The first equation is the bending equation, and the second is the shear force equation. The second boundary condition applied to the second equation yields the result that

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$$\mathrm{GK}\,(v' - \theta) + P\theta = 0 \tag{7}$$

This equation can be solved for the shear strain measure $v' - \theta$, which can then in turn be eliminated from the bending equation. The resulting bending equation and both of its boundary conditions may then be written as

$$\theta'' + \frac{P(P - \mathrm{GK})}{\mathrm{GKEI}}\theta = 0 \qquad \theta(0) = \theta'(\ell) = 0 \tag{8}$$

The solution is $\theta = A\sin(\beta x)$ where A is an arbitrary constant and $\beta\ell = \pi/2$ such that

$$\beta^2 = \frac{P(P - \mathrm{GK})}{\mathrm{GKEI}} = \frac{\pi^2}{4\ell^2} \tag{9}$$

This is a quadratic equation in P, the roots of which can be written as

$$P_{\mathrm{cr}} = \frac{\mathrm{GK}\ell^2 \pm \sqrt{\mathrm{GK}\ell^2[\mathrm{GK}\ell^2 + \pi^2\mathrm{EI}]}}{2\ell^2} \tag{10}$$

A simpler expression for the two roots can be written if we introduce the Euler column buckling load P_0, given by

$$P_0 = \frac{\pi^2\mathrm{EI}}{4\ell^2} \tag{11}$$

In order for either root to be of practical significance, the maximum prebuckling strain (P/EA) must be small compared with unity. This is consistent with the strain energy used, which was derived based on the assumption of small strain. This approximation leads to a somewhat simpler expression for the two roots, given as

$$\frac{P_{\mathrm{cr}}}{\mathrm{GK}} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4P_0}{\mathrm{GK}}} \tag{12}$$

The usual Euler buckling load is found by taking the minus sign and assuming $P_0 < \mathrm{GK}$. The result, with the first-order correction for transverse shear flexibility, is then

$$P_{\mathrm{cr}} = -P_0\left(1 - \frac{P_0}{\mathrm{GK}}\right) \tag{13}$$

Thus, the magnitude of the compressive buckling load is reduced by transverse shear deformation. This correction is of the same order as the change to the fundamental bending frequency caused by transverse shear flexibility and rotary inertia and may be safely ignored for beams that are sufficiently slender. However, since GK can assume values that are much smaller than EA in the case of composite beams, the reduction of the buckling load caused by the P_0/GK term, though small, may not be negligibly small for composite materials in which $G \ll E$.

The other root is unique to beams with high transverse shear flexibility and is simply

$$P_{\mathrm{cr}} = \mathrm{GK} + P_0\left(1 - \frac{P_0}{\mathrm{GK}}\right) \approx \mathrm{GK} \tag{14}$$

Note the sign: This is a tensile force, so that *beams with transverse shear flexibility can actually "buckle" in tension*. A practical illustration of this effect can be found in Kelly [1]. It should be noted that small prebuckling strain for this case actually requires $\mathrm{GK} \ll \mathrm{EA}$. A similar phenomenon may also be observed when two or more parallel beams are designed function together as one shear-flexible beam, such as the dual-beam torque-tube configurations found in certain bearingless helicopter rotor systems.

It is important to note that the quadratic equation is only obtained in the energy approach if the quadratic terms in the longitudinal strain are based on a consistent approximation of the generalized strain measures derived from the geometrically exact equilibrium equations of Reissner [4]. This affects the quadratic terms multiplying P in Eq. (5). If instead these terms are replaced by v'^2 as would be the case for using the stretching as the longitudinal strain, then there is no quadratic equation for P, and the only root is thus the corrected, compressive Euler load. It should also be noted that the identical result for the tensile buckling load can be obtained from the geometrically exact equilibrium equations.

The buckling mode for θ is the same for both modes of buckling and is simply

$$\theta = \sin\left(\frac{\pi x}{2\ell}\right) \tag{15}$$

while the displacement is

$$\frac{v}{\ell} = \frac{4}{\pi}\left(1 - \frac{P_{\mathrm{cr}}}{\mathrm{GK}}\right)\sin^2\left(\frac{\pi x}{4\ell}\right) \tag{16}$$

As an example, we consider the composite beam of Hodges et al. [6] with $\mathrm{GK}\ell^2 = 95.394\mathrm{EI}$, so that $P_0 = 0.025865\mathrm{GK}$. For this case, plots of the two buckling mode shapes for v are shown in Figs. 1 and 2. The first is the usual compressive mode shape, only modestly affected by shear deformation. The second is very different, with relatively small magnitude and opposite sign. The reason for this is that the coefficient $1 - P_{\mathrm{cr}}/\mathrm{GK}$ is approximately equal to unity for the compression buckling case, where $P_{\mathrm{cr}} \approx P_0$ and P_0/GK is small compared with unity; on the other hand the coefficient is of the order of P_0/GK for the tension buckling case, for which P_{cr} is given by Eq. (14).

It is far more insightful to plot the mode shapes on a combined figure showing both the displacement and section rotation together. Figure 3 shows a combined displacement and rotation mode shape for the compression load, a mode where bending and shear combine to produce a positive displacement, and where bending is dominant; while Fig. 4 shows a combined displacement and rotation mode shape for the tension load where bending and shear oppose one another to produce a net negative displacement where transverse

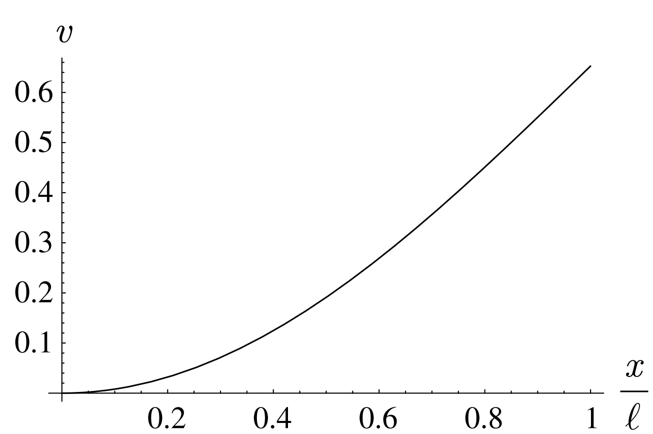

Fig. 1 Displacement mode shape for buckling of a shear-flexible beam in compression.

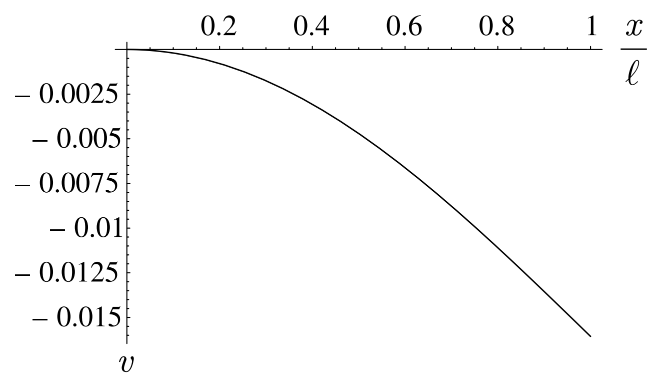

Fig. 2 Displacement mode shape for buckling of a shear-flexible beam in tension.

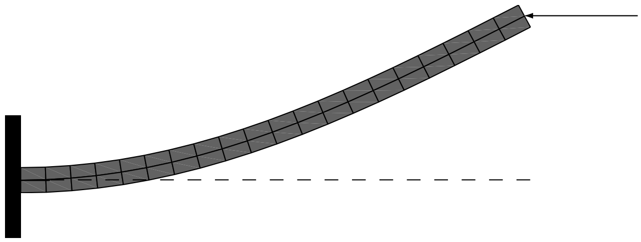

Fig. 3 Combined displacement and rotation mode shape for buckling of a shear-flexible beam in compression.

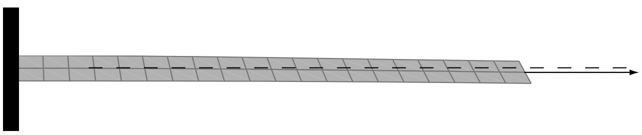

Fig. 4 Combined displacement and rotation mode shape for buckling of a shear-flexible beam in tension.

shear is dominant. Recall that for certain boundary conditions the free-vibration modes and frequencies of Timoshenko beams have a high-frequency spectrum dominated by shearing, while the low-frequency spectrum is dominated by bending [7]; the similarity between the two types of vibration modes and the two types of buckling modes in the present results cannot be coincidental.

Mechanical Analog

A simplified mechanical analog of this configuration is shown in Fig. 5. The rotational spring at the top of the diagram has spring constant of k_1 and is analogous to the beam bending stiffness. The spring at the bottom, with spring constant k_2 is analogous to transverse shearing stiffness in the beam. The two roots have the same form as those of the beam, namely

$$P = \frac{\ell k_2}{2}\left(1 \pm \sqrt{1 + \frac{4k_1}{k_2 \ell^2}}\right) \tag{17}$$

If we regard $k_1 \ll k_2\ell^2$, then the two roots become

$$P = \frac{k_2\ell}{2}\left[1 \pm \left(1 + \frac{2k_1}{k_2\ell^2} - \frac{2k_1^2}{k_2^2\ell^2} + \dots\right)\right] = \begin{cases} -\frac{k_1}{\ell} + \dots \\ k_2\ell + \dots \end{cases} \tag{18}$$

The first root is negative and is a slight perturbation of the buckling load when k_2 is infinite. The second is the tension buckling load for the mechanical analog.

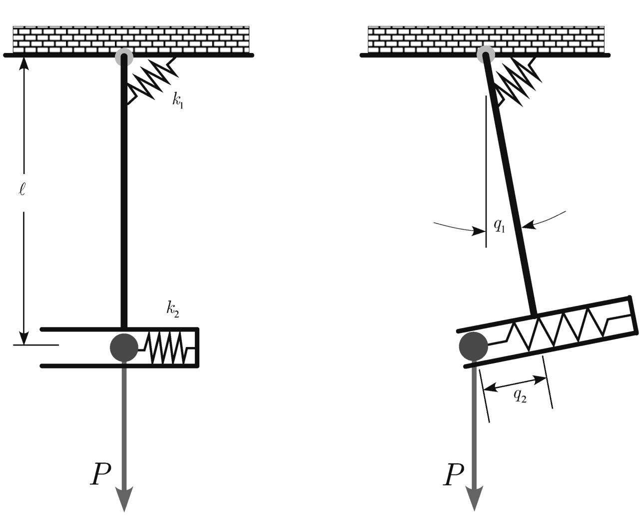

Fig. 5 Schematic of mechanical analog of bending-shear instability.

Shear Buckling Under Tension in Rotating Beams

Results recently obtained [6] show that the tension buckling phenomenon also carries over for rotating beams. It is well known [8] that in a uniform, rotating beam, directed radially outward, with $x = 0$ at the center of rotation, the axial force is

$$P = \frac{\mu\Omega^2}{2}\left(\frac{\ell^2 - x^2}{2}\right) \tag{19}$$

where Ω is the angular speed of the rotation and μ is the mass per unit length. Furthermore, if we only consider static displacement in the plane of rotation, then the kinetic energy per unit length is not zero but is, instead, given by

$$K = \tfrac{1}{2}\mu\Omega^2 v^2 \tag{20}$$

Thus, the Lagrangian $K - U - V$ can be written as

$$K - U - V = \frac{1}{2}\int_0^\ell \left[\mu\Omega^2\left(\frac{\ell^2 - x^2}{2}\right)(\theta^2 - 2v'\theta) + \mu\Omega^2 v^2 - \mathrm{EI}\theta'^2 - \mathrm{GK}(v' - \theta)^2\right]\mathrm{d}x \tag{21}$$

Letting $\Gamma = \mathrm{GK}\ell^2/\mathrm{EI}$, $\lambda^2 = \mu\ell^4\Omega^2/\mathrm{EI}$, $\xi = x/\ell$, and $(\,)' = \ell\,\mathrm{d}(\,)/\mathrm{d}x$, a nondimensional Lagrangian Φ can be defined as

$$\Phi = \frac{\lambda^2}{2}\int_0^2 \left[\left(\frac{1 - \xi^2}{2}\right)(\theta^2 - 2v'\theta) + v^2\right]\mathrm{d}\xi - \frac{1}{2}\int_0^1 [\theta'^2 + \Gamma(v' - \theta)^2]\,\mathrm{d}\xi \tag{22}$$

A simple approximation for the critical angular speed can be found by the use of the Rayleigh–Ritz method. It can be shown that the critical value of λ^2_{cr} tends to infinity as Γ becomes large, that is, where shear deformation is suppressed [9], and that λ^2_{cr} is roughly proportional to 2Γ when $\Gamma \gg 1$. For current rotor systems $\Gamma \gg 1$ and is of order 10^3 to 10^4, but the maximum value of λ^2 is only of the order 10^2. Thus, the angular speed is nowhere near being high enough to induce this kind of instability. However, for composite blades, the value of Γ can be considerably less. For example, consider the composite beam of Hodges et al. [6] having $\Gamma = 95.394$, specialized for the inextensible case. Assuming cubic power series for each of v and θ, one obtains the value $\lambda^2_{\mathrm{cr}} = 210.16$, while a converged beam finite element solution yields $\lambda^2_{\mathrm{cr}} = 209.39$. The present Note is *not* claiming that the phenomenon of tensile buckling of composite beams is of practical significance—only that it is realizable when the parameters have certain values.

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